JOURNAL OF MATHEMATICAL PHYSICS **48**, 113502 (2007)

Extended double-complex linear systems and new multiple infinite-dimensional hidden symmetries for the general symplectic gravity models

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(Received 2 June 2007; accepted 3 October 2007; published online 6 November 2007)

By using a so-called extended double (ED)-complex method, the previously found doubleness symmetry for each member of the class of general symplectic gravity models is further exploited and extended. A $2(n+1)\times 2(n+1)$ matrix double-complex H-potential is constructed for any non-negative integer n, and the motion equations in two dimensions are written in a double-complex formulation. A double-duality mapping is proposed and two pairs of ED-complex Hauser-Ernst-type linear systems [J. Math. Phys. **21**, 1126 (1980)] are established. Based on these linear systems, explicit formulations of new multiple hidden symmetry transformations for the studied theories are given. For any fixed n, these symmetry transformations are verified to constitute multiple infinite-dimensional Lie algebras, each of which is a semidirect product of the Kac-Moody $sp(2(\widehat{n+1}),R)$ and Virasoro algebras (without center charges). These results demonstrate that the ED-complex method is necessary and more effective, and the general symplectic gravity models under consideration possess much richer symmetry structures than previously expected. © *2007 American Institute of Physics.*
[DOI: 10.1063/1.2801880]

I. INTRODUCTION

In the recent past, the studies of symmetries for the dimensionally reduced low energy effective (super)string theories have attracted much attention because of their importance in theoretical and mathematical physics (see, e.g., Refs. 1–17). Such effective string theories describe various interacting matter fields coupled to gravity. The Einstein-Maxwell-dilaton-axion (EMDA) theory[3,8,9,11,14,15] is a typical and important model of this kind. In Ref. 18, Kechkin and Yurova developed a series of symplectic gravity models, each of them is a generalization of the EMDA theory so that it describes a coupled system of n Abelian vector fields and the symmetric $n\times n$ matrix extensions of the dilaton and Kalb-Ramond fields for $n=1,2,\ldots$. We call these general symplectic gravity models "SGM-n" theories for brevity. Thus the EMDA theory is the case of SGM-1. Some symmetries of the SGM-n theories have been found and some analogies between the SGM-n $(n\geqslant 1)$ and the reduced vacuum Einstein theories have been noted. However, many *scalar* functions in pure gravity correspond, formally, to *matrix* ones in the SGM-n theories, thus the noncommuting property of the matrices gives rise to essential complications for the further study of the latter. Moreover, some particular relations, such as for any 2×2 matrix A: $A^{\mathsf{T}}\epsilon A=(\det A)\epsilon$, $A^{\mathsf{T}}\epsilon+\epsilon A=(\operatorname{tr} A)\epsilon$ [with $\epsilon=\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$], have no general analogs for higher dimensional $m\times m$ $(m\geqslant 3)$ matrices, while these relations are important and useful in some studies of the reduced vacuum gravity (e.g., Refs. 19–23). Since in the investigations of the SGM-n $(n\geqslant 1)$

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theories we mainly deal with $2(n+1)\times 2(n+1)$ matrix functions, some deeper researches and further extended studying methods are needed.

The present paper is a continuation of our previous paper.[24] In this paper, by using a so-called extended double (ED)-complex function method,[25] the previously found doubleness symmetry[24] of the stationary axisymmetric (SAS) SGM-n theories is further exploited and extended. Double-complex $2(n+1)\times 2(n+1)$ matrix H-potentials are constructed and the motion equations are extended into double-complex form in terms of these H-potentials. Moreover, we further find that the theories under consideration possess more double symmetries so that, for each n, a double-duality mapping can be introduced and two pairs of ED-complex Hauser-Ernst (HE)-type linear systems can be established. Based on these linear systems, new infinitesimal multiple symmetry transformations for the SGM-n theories are explicitly constructed. Then these symmetry transformations are verified to constitute some multiple infinite-dim Lie algebras. For each fixed n, one of these multiple Lie algebras is a semidirect product of the Kac-Moody $sp(2\widehat{(n+1)},R)$ and Virasoro algebras (without center charges). These results demonstrate that the ED-complex method is necessary and more effective, and the theories under consideration possess much richer symmetry structures than previously expected.

In Sec. II, some related concepts and notations of the ED-complex functions[25] and the double-complex matrix Ernst formulation of the SGM-n field equations[24] are briefly recalled. In Sec. III, double-complex H-potentials are constructed and a pair of ED-complex HE-type linear systems are established for each n. In Sec. 4, by virtue of these linear systems, we give explicit expressions of some infinitesimal double transformations for the studied theories and then verify that these transformations are all double hidden symmetries leaving the SGM-n motion equations and related conditions invariant. The double infinite-dim Lie algebra structures of these hidden symmetries are calculated out in Sec. V. In Sec. VI, to each fixed n, a double-duality mapping is introduced, another pair of ED-complex HE-type linear systems are established, and the associated infinitesimal double symmetry transformations are explicitly given, these constitute another double infinite-dim symmetry Lie algebras of the studied theories. Finally, Sec. VII gives some summary and discussions.

II. ED-COMPLEX FUNCTION AND DOUBLE-COMPLEX MATRIX ERNST EQUATIONS OF THE SGM-*n* THEORIES

For later use, here we briefly recall some related concepts and notations of the ED-complex function[25] and the double-complex matrix Ernst formulation of the SGM-n field equations.[24]

A. ED-complex function[25]

Let i and J denote, respectively, the ordinary and the ED imaginary unit. We shall concern ourselves mainly with some special values of J, i.e., $J=j(j^2=-1,j\neq\pm i)$ or $J=\varepsilon(\varepsilon^2=+1,\varepsilon\neq\pm 1)$. If a series $\Sigma_{n=0}^{\infty}|a_n|,a_n\in\mathbf{C}$ (ordinary complex number) is convergent, then $a(J)=\Sigma_{n=0}^{\infty}a_nJ^{2n}$ is called an ED ordinary complex number, which can correspond to a pair (a_C,a_H) of ordinary complex number, where $a_C:=a(J=j)$, $a_H:=a(J=\varepsilon)$. When $a(J)$ and $b(J)$ both are ED ordinary complex numbers,

$$c(J)=a(J)+Jb(J) \tag{2.1}$$

is called an ED-complex number, it can correspond to a pair (c_C,c_H), where $c_C:=c(J=j)=a_C+jb_C$, $c_H:=c(J=\varepsilon)=a_H+\varepsilon b_H$. If $a(J)$ and $b(J)$ are real, we call them double real and call the corresponding $c(J)$ simply a double-complex number.

We would like to point out that, from the above definitions, J should be taken as an indeterminate rather than a discrete variable. The ED-complex method can be regarded as some "deformation" theory, in which J plays the role of "deformation parameter" (or analytical link, cf. Ref. 26 for nonextended case). By doubleness symmetry we, in fact, mean the symmetry property of the considered theory under this deformation. We call it an ED-complex method only because in most of its applications (e.g., in the present paper) we are mainly interested in the cases of $J=j$ and $J=\varepsilon$.

All ED-complex numbers with usual addition and multiplication constitute a commutative ring. Corresponding to the two imaginary units J and i in this ring, we have two complex conjugations: ED-complex conjugation "$\star$" and ordinary complex conjugation "$-$,"

$$c(J)^\star := a(J) - Jb(J), \quad \overline{c(J)} := \overline{a(J)} + J\overline{b(J)}. \tag{2.2}$$

These imply that $J^\star=-J$, $\bar{J}=J$, $i^\star=i$, $\bar{i}=-i$. If $a(J)$ and $b(J)$ are ED ordinary complex functions of some ordinary complex variables $z_1,\ldots,z_n$, then $c(z_1,\ldots,z_n;J)=a(z_1,\ldots,z_n;J)+Jb(z_1,\ldots,z_n;J)$ is called an ED-complex function. We say $c(z_1,\ldots,z_n;J)$ to be continuous, analytical, etc., if $a(z_1,\ldots,z_n;J)$ and $b(z_1,\ldots,z_n;J)$ both, as ordinary complex functions, have the same properties. We also need ED-complex (function) matrices, and for an ED-complex matrix $W(J)$, we define

$$W(J)^\dagger := [W(J)^\star]^{\mathsf{T}}, \tag{2.3}$$

"T" denotes the transposition. The ED imaginary unit commutation operator "$\circ$" is defined as

$$\circ: J \to \mathring{J}, \quad \mathring{J} = \epsilon, \quad \mathring{\epsilon} = j. \tag{2.4}$$

Obviously, $\mathring{J}$ is the ED imaginary unit, too.

B. Double-complex matrix Ernst formulation of the SGM-n field equations[24]

The actions of class of SGM-n theories in four-dim are[18]

$$S = \int \left\{-R + \mathrm{Tr}\left[\tfrac{1}{2}(\partial p p^{-1})^2 - pFF^{\mathsf{T}} + \tfrac{1}{3}(pH)^2\right]\right\}\sqrt{-g}d^4x, \tag{2.5}$$

where $g_{\mu\nu}$ is the metric (signature $+ - - - -$, $\mu,\nu=0,1,2,3$), R is the Ricci scalar, $g=\det(g_{\mu\nu})$, p is a symmetric $n\times n$ matrix with scalar field components (for the EMDA case, $p=e^{-2\phi}$, ϕ is the dilaton field), and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad H_{\mu\nu\lambda} = \partial_\mu B_{\nu\lambda} - \tfrac{1}{2}(A_\mu F^{\mathsf{T}}_{\nu\lambda} + F_{\nu\lambda}A^{\mathsf{T}}_\mu) + \text{cyclic}, \tag{2.6}$$

in which A_μ is an $n\times 1$ column of Abelian vector fields and $B_{\mu\nu}$ is an extension of the usual Kalb-Ramond tensor field, here each Lorentz component of $B_{\mu\nu}$ $(\mu,\nu=0,1,2,3)$ is a symmetric $n\times n$ matrix and among these matrices there are relations $B_{\mu\nu}=-B_{\nu\mu}$. The SGM-n action (2.5) gives the pure Einstein and the EMDA theories, respectively, when $n=0$ and $n=1$ and provides their generalization for an arbitrary non-negtive integer n.

Now we consider the SAS case, in which the 4 dim space-time metric can be written as[27]

$$ds^2 = f_{AB}dx^A dx^B - e^\gamma \delta_{LN} dx^L dx^N \quad (A,B=0,1,L,N=2,3), \tag{2.7}$$

and f_{AB} can be parametrized as

$$f_{AB} = \begin{pmatrix} f & -f\omega \\ -f\omega & f\omega^2 - \rho^2 f^{-1} \end{pmatrix}. \tag{2.8}$$

After reducing to the SAS case, in addition to the above metric variables, the set of SGM-n dynamical quantities contains two Lorentzian components A_0,A_1 of the $n\times 1$ column four-potential A_μ, one nontrivial Lorentzian component B_{01} of the $n\times n$ matrix extended Kalb-Ramond field $B_{\mu\nu}$, and the $n\times n$ symmetric matrix p of the Lorentzian scalar fields. And all of these fields are assumed to depend only on x^2, x^3. For simplicity, we denote x^2, x^3 by x, y in the following. In terms of these, the essential dynamical equations of the SAS SGM-n theory can be written as[18]

$$d(\rho^{-1}P^{*}dQP)=0, \quad d(\rho^{*}dPP^{-1}+\rho^{-1}P^{*}dQPQ)=0, \tag{2.9}$$

and $\rho=\rho(x,y)>0$ is a harmonic function in 2 dim $\{x,y\}$. Where the notations of differential form are adopted, " *" is the dual operation of 2 dim Euclidian space, the $(n+1)\times(n+1)$ symmetric real matrices P and Q are defined by

$$P=\begin{pmatrix} f-2A_0^{\mathsf{T}}pA_0 & -\sqrt{2}A_0^{\mathsf{T}}p \\ -\sqrt{2}pA_0 & p \end{pmatrix},$$

$$Q=\begin{pmatrix} \omega & -\sqrt{2}(A_1+\omega A_0)^{\mathsf{T}} \\ -\sqrt{2}(A_1+\omega A_0) & (A_1+\omega A_0)A_0^{\mathsf{T}}+A_0(A_1+\omega A_0)^{\mathsf{T}}-B_{01} \end{pmatrix}. \tag{2.10}$$

Thus, if P and Q are known, we can directly obtain the original fields f, ω, A_0, A_1, B_{01}, and p. Moreover, according to the Einstein equations,[18,27] the field $\gamma(x,y)$ in (2.7) can be obtained by a simple integration provided P, Q are known, so we shall focus our attention on Eqs. (2.9).

As pointed out in Ref. 24, the SAS SGM-n theories possess a so-called doubleness symmetry such that the Eqs. (2.9) can be extended into a double-complex matrix Ernst-like formulation,

$$\rho^{-1}d(\rho^{*}dE(J))=dE(J)P(J)^{-1*}dE(J), \tag{2.11}$$

where $E(J)=P(J)+JQ(J)$ [with $P(J)$, $Q(J)$ both being double-real $(n+1)\times(n+1)$ symmetric matrices] is called a matrix double-complex Ernst-like potentials of the SAS SGM-n theories, and the wedge symbol "$\wedge$" in exterior products of differential forms is omitted for simplicity. If a solution $E(J)$ of Eq. (2.11) is known, we can obtain a pair of real solutions of the SAS SGM-n theory.

III. DOUBLE-COMPLEX *H*-POTENTIALS AND ED-COMPLEX HE-TYPE LINEAR SYSTEMS

We introduce a double-real $2(n+1)\times 2(n+1)$ matrix function $M(J)=M(x,y;J)$, which satisfies

$$M(J)^{\mathsf{T}}=M(J), \tag{3.1a}$$

$$M(J)\eta M(J)=J^2\rho^2\eta, \tag{3.1b}$$

$$\eta:=\begin{pmatrix} 0 & I_{n+1} \\ -I_{n+1} & 0 \end{pmatrix}, \tag{3.1c}$$

where I_{n+1} is the $(n+1)$-dim unit matrix. Taking the Gauss decomposition of $M(J)$ as

$$M(J)=\begin{pmatrix} P(J) & -P(J)Q(J) \\ -Q(J)P(J) & Q(J)P(J)Q(J)+J^2\rho^2P(J)^{-1} \end{pmatrix}, \tag{3.2}$$

and noticing (2.4), Eq. (2.11) can be equivalently written as

$$d(\rho^{-1}M(J)\eta^{*}dM(J))=0. \tag{3.3}$$

According to the spirit of Ref. 24, if a solution of Eq. (3.3) with conditions (3.1a), (3.1b), and (3.1c) is known, then by the decomposition (3.2), we can obtain real solutions of the SGM-n theory *in pairs* as follows:

$$(P,Q)=(P_C,Q_C), \tag{3.4a}$$

$$(\hat{P},\hat{Q}) = (T(P_H), V_{P_H}(Q_H)), \tag{3.4b}$$

where the transformations T, V are defined by

$$T{:}P \to T(P) = \rho P^{-1},$$

$$V{:}P,Q \to V_P(Q) = \int \rho^{-1}P(\partial_y Q)Pdx - \rho^{-1}P(\partial_x Q)Pdy, \tag{3.5}$$

and the existence of $V_{P_H}(Q_H)$ is ensured by the $J=\epsilon$ case of Eq. (3.3). Obviously, Eqs. (3.3) and (3.1) are invariant under the global transformations: $M(J)\to G^{\mathsf{T}}M(J)G$, $G\in Sp(2(n+1),R)$. Of course, also as will be seen in the following, the symmetry structures of the considered theories are very much richer than this.

Equation (3.3) implies that we can introduce a double-real $2(n+1)\times 2(n+1)$ matrix twist potential $N(J)=N(x,y;J)$ such that

$$dN(J) = -\rho^{-1}M(J)\eta^{*}dM(J). \tag{3.6}$$

Using (3.1), we obtain from (3.6),

$$dM(J) = -\rho^{-1}J^2M(J)\eta^{*}dN(J). \tag{3.7}$$

Now introducing a $2(n+1)\times 2(n+1)$ matrix double-complex H-potential,

$$H(J) \coloneqq M(J) + JN(J), \tag{3.8}$$

and denoting $\Omega \coloneqq J\eta$, then Eqs. (3.6) and (3.7) can be equivalently written as a single double-complex matrix equation,

$$dH(J) = -\rho^{-1}M(J)\Omega^{*}dH(J). \tag{3.9}$$

Furthermore, from (3.1) and (3.6) we have $d(N(J)-N(J)^{\mathsf{T}})=-2J^{2^*}d\rho\,\eta$. Thus, from the harmony of $\rho(x,y)$, we can introduce another real field $z=z(x,y)$ such that ${}^{*}d\rho=dz$ and obtain $N(J)-N(J)^{\mathsf{T}}=-2J^2z\eta$. These relations and Eqs. (3.1), (3.8), and (2.3) imply that we can express Eq. (3.9) as

$$2(z+\rho^{*})dH(J) = (H(J)+H(J)^{\dagger})\Omega dH(J), \tag{3.10}$$

with (3.1) this is equivalent to (3.3). In addition, from (3.10) we can obtain

$$dH(J)^{\dagger}\Omega dH(J) = dH(J)^{\dagger}\Omega^{*}dH(J) = 0. \tag{3.11}$$

Now we introduce an ordinary complex parameter t and define

$$A(t;J) \coloneqq I - t[H(J)+H(J)^{\dagger}]\Omega \quad [I \text{ is the } 2(n+1)\text{-dim unit matrix}], \tag{3.12}$$

$$\Gamma(t;J) \coloneqq t\Lambda(t)^{-1}dH(J), \tag{3.13}$$

$$\Lambda(t) \coloneqq 1-2t(z+\rho^{*}), \quad \Lambda(t)^{-1} = \lambda(t)^{-2}[1-2t(z-\rho^{*})], \tag{3.14}$$

$$\lambda(t) \coloneqq [(1-2zt)^2+(2\rho t)^2]^{1/2}, \tag{3.15}$$

then Eq. (3.10) can be rewritten as

$$tdH(J) = A(t;J)\Gamma(t;J). \tag{3.16}$$

From Eqs. (3.11), (3.12), and (3.16), we can obtain $d\Gamma(t;J)=\Gamma(t;J)\Omega\Gamma(t;J)$, this is just the complete integrability condition of the following ED-complex linear differential equation,

$$dF(t;J) = \Gamma(t;J)\Omega F(t;J), \tag{3.17}$$

where $F(t;J)=F(x,y,t;J)$ is a $2(n+1)\times 2(n+1)$ ED-complex matrix function of x, y, and t.

Equation (3.17) does not define $F(t;J)$ uniquely, so we suppress some subsidiary conditions consistent with above equations and the requirement that $F(t;J)$ be holomorphic in a neighborhood of $t=0$. From (3.16) and (3.17), and the relation $2t\Lambda^{-1}dz=-\lambda(t)^{-1}d\lambda(t)$, we have

$$dF(0;J) = 0, \quad d[\dot{F}(0;J) - H(J)\Omega F(0;J)] = 0,$$

$$d[\lambda(t)F(t;J)^{\mathsf{T}}\Omega F(t;J)] = 0, \quad d[F(t;J)^{\dagger}\Omega A(t;J)F(t;J)] = 0,$$

where $\dot{F}(t;J) \coloneqq \partial F(t;J)/\partial t$ and the ED-Hermitian conjugation "†" is defined by (2.3). These equations and (3.17) determine $F(t;J)$ up to right multiplication by an arbitrary nondegenerate $2(n+1)\times 2(n+1)$ matrix function of t, so we can use this freedom and choose the integral constants consistently such that

$$F(0;J) = I, \tag{3.18a}$$

$$\dot{F}(0;J) = H(J)\Omega, \tag{3.18b}$$

$$\lambda(t)F(t;J)^{\mathsf{T}}\Omega F(t;J) = \Omega, \tag{3.19a}$$

$$F(t;J)^{\dagger}\Omega A(t;J)F(t;J) = \Omega. \tag{3.19b}$$

We call Eqs. (3.17)–(3.19) an ED-complex HE-type linear system of the SGM-n theories.

Besides, we can establish another ED-complex linear system of the SGM-n theories. Now, for another ordinary complex parameter w, we define

$$\widetilde{A}(w;J) \coloneqq w - (H(J) + H(J)^{\dagger})\Omega, \tag{3.20}$$

$$\widetilde{\Gamma}(w;J) \coloneqq \widetilde{\Lambda}(w)^{-1}dH(J), \tag{3.21}$$

$$\widetilde{\Lambda}(w) \coloneqq w - 2(z+\rho^{*}), \quad \widetilde{\Lambda}(w)^{-1} = \widetilde{\lambda}(w)^{-2}[w - 2(z-\rho^{*})], \tag{3.22}$$

$$\widetilde{\lambda}(w) \coloneqq [(w-2z)^{2} + (2\rho)^{2}]^{1/2}. \tag{3.23}$$

Then Eq. (3.10) can be rewritten as

$$dH = \widetilde{A}(w;J)\widetilde{\Gamma}(w;J), \tag{3.24}$$

by derivations similar to the above, we have

$$d\widetilde{F}(w;J) = \widetilde{\Gamma}(w;J)\Omega\widetilde{F}(w;J), \tag{3.25}$$

and require consistently that $\widetilde{F}(w;J)$ be analytic around $w=0$ and satisfy

$$\widetilde{\lambda}(w)\widetilde{F}(w;J)^{\mathsf{T}}\Omega\widetilde{F}(w;J) = \Omega, \tag{3.26a}$$

$$\widetilde{F}(w;J)^{\dagger}\Omega\widetilde{A}(w;J)\widetilde{F}(w;J) = \Omega, \tag{3.26b}$$

where $\widetilde{F}(w;J)=\widetilde{F}(x,y,w;J)$ is another ED-complex $2(n+1)\times 2(n+1)$ matrix function of x, y, and w.

IV. PARAMETRIZED DOUBLE SYMMETRY TRANSFORMATIONS

By virtue of solutions $F(t;J)$, $\widetilde{F}(w;J)$ of linear systems (3.17)–(3.19) and (3.25) and (3.26), we can explicitly construct parametrized double symmetry transformations for the SGM-n theories. At first, from definitions (3.8), (3.12)–(3.15), and (3.20)–(3.23), we may consistently choose the ED-complex matrix functions $F(t;J)$ and $\widetilde{F}(w;J)$ as

$$\overline{F(t;J)} = F(\bar{t};J), \quad \overline{\widetilde{F}(w;J)} = \widetilde{F}(\bar{w};J) \tag{4.1}$$

[i.e., the ED real and imaginary parts of $F(t;J)$ and $\widetilde{F}(w;J)$ are double ordinary real when t and w are real] in order to ensure the reality of $M(J)$ and $N(J)$ in the transformed $H(J)$. We shall take this choice in the following.

We consider the following infinitesimal double transformation $\delta=\delta(l)$ of potential $H(J)$:

$$\delta H(J) = J^2\frac{1}{l}[F(l;J)TF(l;J)^{-1} - T]\Omega, \tag{4.2}$$

where l is a (finite) real parameter, $F(l;J)$ is a solution of (3.17)–(3.19) with t being replaced by l, $T=T_a\alpha^a \in sp(2(n+1),R)$ [the Lie algebra of the symplectic group $Sp(2(n+1),R)$], T_a are generators of $sp(2(n+1),R)$, and α^a are infinitesimal real constants. Thus we have relation

$$T^{\mathsf{T}}\Omega + \Omega T = 0. \tag{4.3}$$

Now we prove that (4.2) is a hidden symmetry transformation of the double-complex SGM-n motion equation (3.10) and conditions (3.1). First, from (4.2), (4.3), and (3.19a), we have

$$\begin{aligned}\delta H(J) - \delta H(J)^{\mathsf{T}} &= J^2\frac{1}{l}[F(l;J)TF(l;J)^{-1} - T]\Omega + J^2\frac{1}{l}\Omega[F(l;J)^{\mathsf{T}-1}T^{\mathsf{T}}F(l;J)^{\mathsf{T}} - T^{\mathsf{T}}]\\ &= J^2\frac{1}{l}F(l;J)[TF(l;J)^{-1}\Omega F(l;J)^{\mathsf{T}-1} + F(l;J)^{-1}\Omega F(l;J)^{\mathsf{T}-1}T^{\mathsf{T}}]F(l;J)^{\mathsf{T}}\\ &= J^2\frac{\lambda(l)}{l}F(l;J)(T\Omega + \Omega T^{\mathsf{T}})F(l;J)^{\mathsf{T}} = 0.\end{aligned} \tag{4.4}$$

From (3.8) and definition of $z(x,y)$, Eq. (4.4) implies $\delta M(J)^{\mathsf{T}}=\delta M(J)$ and $\delta z=0$.

In addition, Eqs. (3.8), (3.12), and (4.4) give $M(J)=(J^2/4l)[A(J)-A(J)^{\star}]\Omega$ and $\delta M(J)=(1/2)[\delta H(J)+\delta H(J)^{\dagger}]$, then from (4.2), (4.3), and (3.19b) we have

$$\begin{aligned}\delta M(J)\Omega M(J) + M(J)\Omega\,\delta M(J) &= \frac{J^2}{8l^2}[(-J^2F(l;J)TF(l;J)^{-1} + \Omega F(l;J)^{\dagger-1}T^{\dagger}F(l;J)^{\dagger}\Omega)(A(J) - A(J)^{\star})\\ &\quad + (A(J) - A(J)^{\star})(-J^2F(l;J)TF(l;J)^{-1} + \Omega F(l;J)^{\dagger-1}T^{\dagger}F(l;J)^{\dagger}\Omega)]\Omega\\ &= \frac{J^2}{4l^2}[-J^2A(J)F(l;J)TF(l;J)^{-1} + J^2F(l;J)TF(l;J)^{-1}A(J)^{\star}\\ &\quad - A(J)^{\star}\Omega F(l;J)^{\dagger-1}T^{\dagger}F(l;J)^{\dagger}\Omega + \Omega F(l;J)^{\dagger-1}T^{\dagger}F(l;J)^{\dagger}\Omega A(J)]\Omega\\ &= \frac{J^2}{4l^2}[\Omega F(l;J)^{\dagger-1}\Omega TF(l;J)^{-1} + \Omega F(l;J)^{\dagger-1}T^{\dagger}\Omega F(l;J)^{-1}\\ &\quad - \lambda(l)^2F(l;J)T\Omega F(l;J)^{\dagger}\Omega - \lambda(l)^2F(l;J)\Omega T^{\dagger}F(l;J)^{\dagger}\Omega]\Omega = 0,\end{aligned} \tag{4.5}$$

where the relations

$$A(J) + A(J)^{\star} = 2(1 - 2lz), \quad A(J)A(J)^{\star} = \lambda(l)^2, \tag{4.6}$$

and $T^{\dagger}=T^{\mathsf{T}}$ in the real Lie algebra $sp(2(n+1),R)$ have been used. Equation (4.5) implies that, under the transformation (4.2), the condition (3.1b) is preserved and $\delta\rho=0$.

Now we investigate the equation satisfied by $\delta H(J)$. From (4.2) and (3.17), it follows that $d(\delta H)=(J^2/l)[\Gamma(l;J)\Omega, F(l;J)TF(l;J)^{-1}]\Omega$, this and (3.13) and (3.10) further followed by

$$2(z+\rho^{*})d(\delta H(J)) = (H(J)+H(J)^{\dagger})\Omega d(\delta H(J)) - \frac{1}{l}[(H(J)+H(J)^{\dagger})\Omega, F(l;J)TF(l;J)^{-1}]\Gamma(l;J). \tag{4.7}$$

On the other hand, from (4.2), (4.3), (3.12), (3.16), and (3.19b) we have

$$(\delta H(J) + \delta H(J)^{\dagger})\Omega dH(J) = \frac{J^2}{l^2}[F(l;J)TF(l;J)^{-1}\Omega + \Omega F(l;J)^{\dagger -1}T^{\dagger}F(l;J)^{\dagger}]\Omega A(l;J)\Gamma(l;J)$$

$$= -\frac{1}{l}[(H(J)+H(J)^{\dagger})\Omega, F(l;J)TF(l;J)^{-1}]\Gamma(l;J).$$

Substituting this into Eq. (4.7), we finally obtain

$$2(z+\rho^{*})d(\delta H(J)) = (H(J)+H(J)^{\dagger})\Omega d(\delta H(J)) + (\delta H(J)+\delta H(J)^{\dagger})\Omega dH(J). \tag{4.8}$$

Equations (4.8), (4.4), and (4.5) show that $H(J)+\delta H(J)$ with $\delta H(J)$ given by (4.2) satisfies the same equation (3.10) and conditions [(3.1a) and (3.1b)] as $H(J)$ does, i.e., (4.2) is indeed a double symmetry transformation for the SAS SGM-n motion equations.

Similarly, by using solution $\widetilde{F}(s;J)$ of (3.25) and (3.26), we can construct another parametrized double infinitesimal symmetry transformation of the SGM-n theory as

$$\widetilde{\delta}H(J) = -J^2 s[\widetilde{F}(s;J)T\widetilde{F}(s;J)^{-1} - T]\Omega, \tag{4.9}$$

where s is a finite real parameter.

The set of symmetry transformations of the SAS EMDA theory can be further enlarged. In addition to (4.2) and (4.9), we propose two other infinitesimal double transformations

$$\Delta H(J) = -J^2\sigma\dot{F}(l;J)F(l;J)^{-1}\Omega, \tag{4.10}$$

$$\widetilde{\Delta}H(J) = J^2\epsilon s[s\widetilde{F}(s;J)\dot{\widetilde{F}}(s;J)^{-1} + \tfrac{1}{2}]\Omega, \tag{4.11}$$

where l, s both are finite real parameters and σ, ϵ are infinitesimal real constants.

From (4.10) and (3.19a),

$$\Delta H(J) - \Delta H(J)^{\mathsf{T}} = -J^2\sigma[\dot{F}(l;J)F(l;J)^{-1}\Omega + \Omega F(l;J)^{\mathsf{T}-1}\dot{F}(l;J)^{\mathsf{T}}] = J^2\sigma\lambda(l)^{-1}\frac{\partial}{\partial l}\lambda(l)\Omega$$

$$= -J^2\frac{2\sigma}{\lambda(l)^2}[z(1-2lz) - 2l\rho^2]\Omega, \tag{4.12}$$

this and the definition of $z(x,y)$ imply $(\Delta M(J))^{\mathsf{T}}=\Delta M(J)$ and $\Delta z=(\sigma/\lambda(l)^2)[z(1-2lz)-2l\rho^2]$.

Moreover, since $M(J)=\frac{1}{2}[H(J)+H(J)^{\star}]$ and $(\Delta M(J))^{\mathsf{T}}=\Delta M(J)$ by (4.12), we have

$$\Delta M(J) = \tfrac{1}{2}[\Delta H(J) + \Delta H(J)^{\star}] = \tfrac{1}{2}[\Delta H(J)^{\dagger} + \Delta H(J)^{\mathsf{T}}] = (\Delta M(J))^{\dagger}, \tag{4.13a}$$

$$[\Delta M(J)\Omega M(J) + M(J)\Omega\Delta M(J)]^{\dagger} = \Delta M(J)\Omega M(J) + M(J)\Omega\Delta M(J). \tag{4.13b}$$

Thus from (4.13a), (3.19b), and (4.6), it follows that

$$
\begin{aligned}
&\Delta M(J)\Omega M(J) + M(J)\Omega\Delta M(J)\\
&= \frac{J^2}{8l}[(\Delta H(J)^\dagger + \Delta H(J)^\mathsf{T})\Omega(A(J) - A(J)^\star) + (A(J) - A(J)^\star)(\Delta H(J)^\dagger + \Delta H(J)^\mathsf{T})\Omega]\Omega\\
&= \frac{J^2}{4l}[\Delta H(J)^\dagger\Omega A(J) - A(J)^\star\Delta H(J)^\dagger\Omega + A(J)\Delta H(J)^\mathsf{T}\Omega - \Delta H(J)^\mathsf{T}\Omega A(J)^\star]\Omega\\
&= \frac{\sigma}{4l}\Bigg[\Omega\frac{\partial}{\partial l}F(l;J)^{\dagger-1}\Omega F(l;J)^{-1}\Omega + \Omega\frac{\partial}{\partial l}F(l;J)^{\mathsf{T}-1}\Omega F(l;J)^{\star-1}\Omega\\
&\quad - \frac{1}{\lambda(l)^2}\left(A(J)^\star\Omega\frac{\partial}{\partial l}F(l;J)^{\dagger-1}\Omega F(l;J)^{-1}A(J)^\star\Omega + A(J)\Omega\frac{\partial}{\partial l}F(l;J)^{\mathsf{T}-1}\Omega F(l;J)^{\star-1}A(J)\Omega\right)\Bigg],
\end{aligned}
$$

then from (4.13b), (3.19b), and (4.6) we obtain

$$
\begin{aligned}
&\Delta M(J)\Omega M(J) + M(J)\Omega\Delta M(J)\\
&= \frac{1}{2}[\Delta M(J)\Omega M(J) + M(J)\Omega\Delta M(J) + (\Delta M(J)\Omega M(J) + M(J)\Omega\Delta M(J))^\dagger]\\
&= \frac{\sigma J^2}{8l}\left[\frac{1}{\lambda(l)^2}\left(A(J)^\star\frac{\partial}{\partial l}A(J)A(J)^\star + A(J)\frac{\partial}{\partial l}A(J)^\star A(J)\right) - \left(\frac{\partial}{\partial l}A(J) + \frac{\partial}{\partial l}A(J)^\star\right)\right]\Omega\\
&= \frac{\sigma J^2}{8l\lambda(l)^2}\left[\frac{\partial}{\partial l}(\lambda(l)^2)(A(J)+A(J)^\star) - 2\lambda(l)^2\frac{\partial}{\partial l}(A(J)+A(J)^\star)\right]\Omega\\
&= \frac{2\sigma}{\lambda(l)^2}J^2\rho^2\Omega. \qquad (4.14)
\end{aligned}
$$

This result shows that the double transformation (4.10) preserves the condition (3.1b) provided $\Delta\rho=(\sigma/\lambda(l)^2)\rho$, and we can also verify, by direct calculations, that ${}^*d(\Delta\rho)=d(\Delta z)$ as desired.

Now we consider the equation satisfied by the transformed fields. From (3.10), (3.13), (3.14), (4.12), and (4.14), we have

$$2(\Delta z + \Delta\rho^*)dH(J) = 2\sigma(z+\rho^*)\Lambda(l)^{-1}dH(J) = \frac{\sigma}{l}(H(J)+H(J)^\dagger)\Omega\Gamma(l;J). \qquad (4.15)$$

Moreover, from (4.10), (3.13), (3.14), and (3.17) we obtain

$$d\Delta H(J) = \sigma\dot{\Gamma}(l;J) - \sigma J^2[\Gamma(l;J)\Omega, \dot{F}(l;J)F(l;J)^{-1}]\Omega. \qquad (4.16)$$

Multiplying (4.16) from left by $2(z+\rho^*)$ and using (3.10) and (4.16) again, it follows that

$$2(z+\rho^*)d\Delta H(J) = \sigma[(H(J)+H(J)^\dagger)\Omega, \dot{F}(l;J)F(l;J)^{-1}]\Gamma(l;J) + (H(J)+H(J)^\dagger)\Omega d\Delta H(J). \qquad (4.17)$$

On the other hand, from (4.10), (3.12), (3.16), and (3.19b) we have

$$
\begin{aligned}
&(\Delta H(J) + \Delta H(J)^\dagger)\Omega dH(J)\\
&= -J^2\sigma l^{-1}[\dot{F}(l;J)F(l;J)^{-1}\Omega + \Omega F(l;J)^{\dagger-1}\dot{F}(l;J)^\dagger]\Omega A(J)\Gamma(l;J)\\
&= \sigma[(H(J)+H(J)^\dagger)\Omega, \dot{F}(l;J)F(l;J)^{-1}]\Gamma(l;J) + \sigma l^{-1}(H(J)+H(J)^\dagger)\Omega\Gamma(l;J). \qquad (4.18)
\end{aligned}
$$

Finally, (4.15), (4.17), and (4.18) give

$$2(\Delta z + \Delta\rho^*)dH(J) + 2(z+\rho^*)d\Delta H(J) = (\Delta H(J)+\Delta H(J)^\dagger)\Omega dH(J) + (H(J)+H(J)^\dagger)\Omega d\Delta H(J). \qquad (4.19)$$

The above results show that (4.10) is indeed a double symmetry transformation of Eq. (3.10) with conditions (3.1a) and (3.1b).

Similarly, we can prove that (4.11), which gives $\tilde{\Delta}z=(\epsilon s/\tilde{\lambda}(s)^2)[z(s-2z)-2\rho^2]$ and $\tilde{\Delta}\rho=(\epsilon s^2/\tilde{\lambda}(s)^2)\rho$, is also a double symmetry transformation of Eqs. (3.10), (3.1a), and (3.1b).

V. INFINITE-DIM ALGEBRA STRUCTURES OF THE DOUBLE SYMMETRIES

From the structures of the double transformations (4.2) and (4.9), we expand the right-hand sides of them in powers of l and s, respectively, as

$$\delta H(J)=\sum_{m=0}^{\infty} l^m \delta^{(m)} H(J), \tag{5.1a}$$

$$\tilde{\delta} H(J)=\sum_{k=1}^{\infty} s^k \tilde{\delta}^{(k)} H(J), \tag{5.1b}$$

where the analytic property of $F(l;F)$, $\tilde{F}(s;J)$ around $l=0$, $s=0$ is noted. Each of $\delta^{(m)}$ and $\tilde{\delta}^{(k)}$ satisfies the same equations and conditions as δ and $\tilde{\delta}$ do, thus we have, in fact, constructed infinite many infinitesimal double hidden symmetry transformations for each SGM-n theory. The algebraic structures of these transformations can be obtained as follows. Noticing the dependence of (4.2) and (4.9) on the parameters l, s and the infinitesimal constants α^a in T, we denote the corresponding transformations by $\delta_\alpha(l)$, $\tilde{\delta}_\alpha(s)$, respectively. Thus we have

$$\begin{aligned}[\delta_\alpha(l),\delta_\beta(l')]H(J) &= J^2\frac{1}{l}[\delta_\beta(l')F(l;J)F(l;J)^{-1},F(l;J)T_\alpha F(l;J)^{-1}]\Omega \\ &\quad - J^2\frac{1}{l'}[\delta_\alpha(l)F(l';J)F(l';J)^{-1},F(l';J)T_\beta F(l';J)^{-1}]\Omega,\end{aligned} \tag{5.2}$$

$$\begin{aligned}[\delta_\alpha(l),\tilde{\delta}_\beta(s)]H(J) &= J^2\frac{1}{l}[\tilde{\delta}_\beta(s)F(l;J)F(l;J)^{-1},F(l;J)T_\alpha F(l;J)^{-1}]\Omega \\ &\quad + J^2 s[\delta_\alpha(l)\tilde{F}(s;J)\tilde{F}(s;J)^{-1},\tilde{F}(s;J)T_\beta \tilde{F}(s;J)^{-1}]\Omega,\end{aligned} \tag{5.3}$$

$$\begin{aligned}[\tilde{\delta}_\alpha(s),\tilde{\delta}_\beta(s')]H(J) &= -J^2 s[\tilde{\delta}_\beta(s')\tilde{F}(s;J)\tilde{F}(s;J)^{-1},\tilde{F}(s;J)T_\alpha \tilde{F}(s;J)^{-1}]\Omega \\ &\quad + J^2 s'[\tilde{\delta}_\alpha(s)\tilde{F}(s';J)\tilde{F}(s';J)^{-1},\tilde{F}(s';J)T_\beta \tilde{F}(s';J)^{-1}]\Omega,\end{aligned} \tag{5.4}$$

where $T_\alpha=\alpha^a T_a$, $\delta(l')F(l;J)=F(l,H(J)+\delta(l')H(J);J)-F(l,H(J);J)$, etc.

To obtain the above commutators explicitly, we need the variations of $F(l;J)$, $\tilde{F}(s;J)$ induced by $\delta(l')H(J)$, $\tilde{\delta}(s')H(J)$. It may be verified by tedious but straightforward calculations that we can take

$$\delta_\alpha(l')F(l;J)=\frac{l}{l-l'}[F(l';J)T_\alpha F(l';J)^{-1}-F(l;J)T_\alpha F(l;J)^{-1}]F(l;J), \tag{5.5}$$

$$\tilde{\delta}_\alpha(s)F(l;J)=\frac{ls}{1-ls}[\tilde{F}(s;J)T_\alpha \tilde{F}(s;J)^{-1}-F(l;J)T_\alpha F(l;J)^{-1}]F(l;J), \tag{5.6}$$

$$\delta_\alpha(l)\tilde{F}(s;J)=\frac{1}{1-ls}[F(l;J)T_\alpha F(l;J)^{-1}-\tilde{F}(s;J)T_\alpha \tilde{F}(s;J)^{-1}]\tilde{F}(s;J), \tag{5.7}$$

$$\widetilde{\delta}_\alpha(s')\widetilde{F}(s;J) = \frac{s'}{s-s'}[\widetilde{F}(s';J)T_\alpha\widetilde{F}(s';J)^{-1} - \widetilde{F}(s;J)T_\alpha\widetilde{F}(s;J)^{-1}]\widetilde{F}(s;J), \tag{5.8}$$

such that $F(l;J)+\delta_\alpha(l')F(l;J)$, $F(l;J)+\widetilde{\delta}_\alpha(s)F(l;J)$ satisfy the same equation (3.17) and conditions (3.18) and (3.19) as $F(l;J)$ does; while $\widetilde{F}(s;J)+\delta_\alpha(l)\widetilde{F}(s;J)$, $\widetilde{F}(s;J)+\widetilde{\delta}_\alpha(s')\widetilde{F}(s;J)$ satisfy the same equation (3.25) and conditions (3.26) as $\widetilde{F}(s;J)$ does.

Substituting (5.5)–(5.8) into (5.2)–(5.4), using (4.2) and (4.9) again and writing $\delta_\alpha(l)H(J) = \alpha^a\delta_a(l)H(J)$, etc., we obtain

$$[\delta_\alpha(l),\delta_\beta(l')]H(J) = \frac{\alpha^a\beta^b}{l-l'}C_{ab}^c(l\delta_c(l)H(J) - l'\delta_c(l')H(J)), \tag{5.9}$$

$$[\delta_\alpha(l),\widetilde{\delta}_\beta(s)]H(J) = \frac{\alpha^a\beta^b}{1-ls}C_{ab}^c(ls\delta_c(l)H(J) + \widetilde{\delta}_c(s)H(J)), \tag{5.10}$$

$$[\widetilde{\delta}_\alpha(s),\widetilde{\delta}_\beta(s')]H(J) = \frac{\alpha^a\beta^b}{s-s'}C_{ab}^c(s'\widetilde{\delta}_c(s)H(J) - s\widetilde{\delta}_c(s')H(J)), \tag{5.11}$$

where C_{ab}^c's are structure constants of the Lie algebra $sp(2(n+1),R)$. Writing (5.1a) and (5.1b) in the explicitly α related forms as

$$\delta_\alpha(l)H(J) = \alpha^a\sum_{m=0}^{\infty} l^m\delta_a^{(m)}H(J), \tag{5.12a}$$

$$\widetilde{\delta}_\alpha(s)H(J) = \alpha^a\sum_{k=1}^{\infty} s^k\widetilde{\delta}_a^{(k)}H(J), \tag{5.12b}$$

and then expanding both sides of (5.9)–(5.11), we finally obtain

$$[\delta_a^{(m)},\delta_b^{(k)}]H(J) = C_{ab}^c\delta_c^{(m+k)}H(J), \quad m,k = 0,\pm 1,\pm 2,\ldots, \tag{5.13}$$

where $\delta_a^{(-k)}H(J) \coloneqq \widetilde{\delta}_a^{(k)}H(J)$ for $k\geqslant 1$. Thus, the infinite set of symmetry transformations $\{\delta_a^{(m)}, m = 0,\pm 1,\pm 2,\ldots\}$ constitute a double affine Kac-Moody $sp(2\widehat{(n+1)},R)$ algebra (without center charge).

Now we consider transformations (4.10) and (4.11). They can be expanded as

$$\Delta H(J) = \sigma\sum_{m=0}^{\infty} l^m\Delta^{(m)}H(J), \tag{5.14a}$$

$$\widetilde{\Delta}H(J) = \epsilon\sum_{k=1}^{\infty} s^k\widetilde{\Delta}^{(k)}H(J). \tag{5.14b}$$

Thus we obtain another infinite set of double symmetry transformations $\{\Delta^{(m)},\widetilde{\Delta}^{(k)}, m = 0,1,2,\ldots; k=1,2,\ldots\}$ of the SGM-n theory. To calculate their commutators, we first denote (4.10) and (4.11) by $\Delta_\sigma(l)H(J)$, $\widetilde{\Delta}_\epsilon(s)H(J)$, respectively, and then have

$$[\Delta_\sigma(l),\Delta_{\sigma'}(l')]H(J) = -J^2\sigma\frac{\partial}{\partial l}(\Delta_{\sigma'}(l')F(l;J)F(l;J)^{-1})\Omega + J^2\sigma'\frac{\partial}{\partial l'}(\Delta_\sigma(l)F(l';J)F(l';J)^{-1})\Omega - J^2\sigma[\Delta_{\sigma'}(l')F(l;J)F(l;J)^{-1},\dot{F}(l;J)F(l;J)^{-1}]\Omega + J^2\sigma'[\Delta_\sigma(l)F(l';J)F(l';J)^{-1},\dot{F}(l';J)F(l';J)^{-1}]\Omega, \tag{5.15}$$

$$[\Delta_\sigma(l),\tilde{\Delta}_\epsilon(s)]H(J) = -J^2\sigma\frac{\partial}{\partial l}(\tilde{\Delta}_\epsilon(s)F(l;J)F(l;J)^{-1})\Omega - J^2\epsilon s^2\frac{\partial}{\partial s}(\Delta_\sigma(l)\tilde{F}(s;J)\tilde{F}(s;J)^{-1})\Omega - J^2\sigma[\tilde{\Delta}_\epsilon(s)F(l;J)F(l;J)^{-1},\dot{F}(l;J)F(l;J)^{-1}]\Omega - J^2\epsilon s^2[\Delta_\sigma(l)\tilde{F}(s;J)\tilde{F}(s;J)^{-1},\dot{\tilde{F}}(s;J)\tilde{F}(s;J)^{-1}]\Omega, \tag{5.16}$$

$$[\tilde{\Delta}_\epsilon(s),\tilde{\Delta}_{\epsilon'}(s')]H(J) = J^2\epsilon s^2\frac{\partial}{\partial s}(\tilde{\Delta}_{\epsilon'}(s')\tilde{F}(s;J)\tilde{F}(s;J)^{-1})\Omega - J^2\epsilon' s'^2\frac{\partial}{\partial s'}(\tilde{\Delta}_\epsilon(s)\tilde{F}(s';J)\tilde{F}(s';J)^{-1})\Omega + J^2\epsilon s^2[\tilde{\Delta}_{\epsilon'}(s')\tilde{F}(s;J)\tilde{F}(s;J)^{-1},\dot{\tilde{F}}(s;J)\tilde{F}(s;J)^{-1}]\Omega - J^2\epsilon' s'^2[\tilde{\Delta}_\epsilon(s)\tilde{F}(s';J)\tilde{F}(s';J)^{-1},\dot{\tilde{F}}(s';J)\tilde{F}(s';J)^{-1}]\Omega. \tag{5.17}$$

As for $\Delta_\sigma(l')F(l;J)$, $\Delta_\sigma(l)\tilde{F}(s;J)$, etc., we propose

$$\Delta_\sigma(l')F(l;J) = \sigma\frac{l}{l-l'}[l\dot{F}(l;J)F(l;J)^{-1} - l'\dot{F}(l';J)F(l';J)^{-1}]F(l;J), \tag{5.18}$$

$$\tilde{\Delta}_\epsilon(s)F(l;J) = \epsilon\frac{ls}{ls-1}\left[l\dot{F}(l;J)F(l;J)^{-1} + s\dot{\tilde{F}}(s;J)\tilde{F}(s;J)^{-1} + \frac{1}{2}\right]F(l;J), \tag{5.19}$$

$$\Delta_\sigma(l)\tilde{F}(s;J) = \sigma\frac{1}{ls-1}\left[s\dot{\tilde{F}}(s;J)\tilde{F}(s;J)^{-1} + l\dot{F}(l;J)F(l;J)^{-1} + \frac{1}{2}\right]\tilde{F}(s;J), \tag{5.20}$$

$$\tilde{\Delta}_\epsilon(s')\tilde{F}(s;J) = \epsilon\frac{s'}{s-s'}[s\dot{\tilde{F}}(s;J)\tilde{F}(s;J)^{-1} - s'\dot{\tilde{F}}(s';J)\tilde{F}(s';J)^{-1}]\tilde{F}(s;J). \tag{5.21}$$

By some lengthy but straightforward calculations, it can be verified that (5.18) and (5.19) are double symmetry transformations of Eq. (3.17) with conditions (3.18) and (3.19); while (5.20) and (5.21) are double symmetry transformations of Eq. (3.25) with conditions (3.26).

Substituting (5.18)–(5.21) into (5.15)–(5.17) and using (4.10) and (4.11), again, it follows that

$$[\Delta_\sigma(l),\Delta_{\sigma'}(l')]H(J) = \sigma\frac{\partial}{\partial l}\left[\frac{l}{l-l'}(l\Delta_{\sigma'}(l)H(J) - l'\Delta_{\sigma'}(l')H(J))\right] - \sigma'\frac{\partial}{\partial l'}\left[\frac{l'}{l'-l}(l'\Delta_\sigma(l')H(J) - l\Delta_\sigma(l)H(J))\right], \tag{5.22}$$

$$[\Delta_\sigma(l),\tilde{\Delta}_\epsilon(s)]H(J) = \sigma\frac{\partial}{\partial l}\left[\frac{ls}{ls-1}(l\Delta_\epsilon(l)H(J) - s^{-1}\tilde{\Delta}_\epsilon(s)H(J))\right] + \epsilon s^2\frac{\partial}{\partial s}\left[\frac{1}{ls-1}(l\Delta_\sigma(l)H(J) - s^{-1}\tilde{\Delta}_\sigma(s)H(J))\right], \tag{5.23}$$

$$[\widetilde{\Delta}_\epsilon(s),\widetilde{\Delta}_{\epsilon'}(s')]H(J)=\epsilon s^2\frac{\partial}{\partial s}\left[\frac{s'}{s-s'}(s^{-1}\widetilde{\Delta}_{\epsilon'}(s)H(J)-s'^{-1}\widetilde{\Delta}_{\epsilon'}(s')H(J))\right]$$
$$-\epsilon' s'^2\frac{\partial}{\partial s'}\left[\frac{s}{s'-s}(s'^{-1}\widetilde{\Delta}_{\epsilon}(s')H(J)-s^{-1}\widetilde{\Delta}_{\epsilon}(s)H(J))\right]. \qquad (5.24)$$

By using (5.14a) and (5.14b) to expand both sides of (5.22)–(5.24), we obtain

$$[\Delta^{(m)},\Delta^{(k)}]H(J)=(m-k)\Delta^{(m+k)}H(J),\quad m,k=0,\pm 1,\pm 2,\ldots, \qquad (5.25)$$

where we have written $\Delta^{(-k)}H(J):=\widetilde{\Delta}^{(k)}H(J)$ for $k\geqslant 1$. This shows that the infinite set of double symmetry transformations $\{\Delta^{(m)},m=0,\pm1,\pm2,\ldots\}$ constitute a double Virasoro algebra (without central charge).

Next we investigate the commutators between the members of $\{\delta^{(m)}\}$ and $\{\Delta^{(k)}\}$. For example, from (4.2), (4.10), (5.5), and (5.18) we have, by some calculations

$$[\Delta_\sigma(l),\delta_a(s)]H(J)=\sigma\frac{\partial}{\partial l}\left[\frac{l}{l-s}(l\delta_a(l)H(J)-s\delta_a(s)H(J))\right]-\sigma\frac{l}{l-s}\frac{\partial}{\partial l}(l\delta_a(l)H(J))$$
$$+\sigma\frac{s}{l-s}\frac{\partial}{\partial s}(s\delta_a(s)H(J)). \qquad (5.26)$$

Similarly, we can give out the expressions of $[\Delta_\sigma(l),\widetilde{\delta}_a(s)]H(J)$, $[\widetilde{\Delta}_\sigma(l),\delta_a(s)]H(J)$, and $[\widetilde{\Delta}_\sigma(l),\widetilde{\delta}_a(s)]H(J)$. Then by using (5.12a), (5.12b), (5.14a), and (5.14b) to expand both sides of these results, we finally obtain

$$[\Delta^{(k)},\delta_a^{(m)}]H(J)=-m\delta_a^{(k+m)}H(J),\quad k,m=0,\pm1,\pm2,\ldots. \qquad (5.27)$$

VI. DOUBLE-DUALITY SYMMETRY AND MULTIPLE INFINITE-DIM SYMMETRY ALGEBRAS

Equations (5.13), (5.25), and (5.27) show that the symmetry transformations (4.2) and (4.9)–(4.11) give a double representation of semidirect product of the affine $sp(\widehat{2n+1},R)$ and Virasoro algebras. These give expression to that the infinite-dim symmetry structures of the SGM-*n* theories contain not only the double Kac-Moody $sp(2\widehat{(n+1)},R)$ algebras but also the double Virasoro algebras. However, the theory under consideration has even more symmetries. To see this, we noted that the Eqs. (3.3) and (3.1) with Gauss decomposition (3.2) imply that for each n there exists a double-duality mapping $D(J)$ as follows:

$$D(J):\ M(J)\mapsto\hat{M}(J)=\begin{pmatrix}\hat{P}(J) & -\hat{P}(J)\hat{Q}(J)\\ -\hat{Q}(J)\hat{P}(J) & \hat{Q}(J)\hat{P}(J)\hat{Q}(J)+J^2\rho^2\hat{P}(J)^{-1}\end{pmatrix}, \qquad (6.1a)$$

$$\hat{P}(J)=T(P(\mathring{J})),\quad \hat{Q}(J)=\mathring{J}^2V_{P(\mathring{J})}(Q(\mathring{J})), \qquad (6.1b)$$

where the transformations T, V, and overcircle operation "o" are defined by (3.5) and (2.4), respectively.

It can be directly verified that $\hat{M}(J)$ satisfies

$$d(\rho^{-1}\hat{M}(J)\eta^* d\hat{M}(J))=0, \qquad (6.2)$$

$$\hat{M}(J)^{\mathsf{T}}=\hat{M}(J), \qquad (6.3a)$$

$$\hat{M}(J)\eta\hat{M}(J) = J^2\rho^2\eta, \tag{6.3b}$$

which are the same in form as (3.3), (3.1a), and (3.1b). Thus, similar to the above discussions, we can obtain the corresponding $\hat{H}(J)$, $\hat{\Gamma}(J)$, $\hat{A}(J)$, $\hat{\tilde{\Gamma}}(J)$, $\hat{\tilde{A}}(J)$, etc., double-complex equation,

$$2(z+\rho^*)d\hat{H}(J) = (\hat{H}(J) + \hat{H}(J)^\dagger)\Omega d\hat{H}(J), \tag{6.4}$$

and a pair of ED-complex HE-type linear systems,

$$d\hat{F}(t;J) = \hat{\Gamma}(t;J)\Omega\hat{F}(t;J), \tag{6.5}$$

$$\hat{F}(0;J) = I, \tag{6.6a}$$

$$\dot{\hat{F}}(0;J) = \hat{H}(J)\Omega, \tag{6.6b}$$

$$\lambda(t)\hat{F}(t;J)^{\mathsf{T}}\Omega\hat{F}(t;J) = \Omega, \tag{6.7a}$$

$$\hat{F}(t;J)^\dagger\Omega\hat{A}(t;J)\hat{F}(t;J) = \Omega, \tag{6.7b}$$

$$d\hat{\tilde{F}}(w;J) = \hat{\tilde{\Gamma}}(w;J)\Omega\hat{\tilde{F}}(w;J), \tag{6.8}$$

$$\tilde{\lambda}(w)\hat{\tilde{F}}(w;J)^{\mathsf{T}}\Omega\hat{\tilde{F}}(w;J) = \Omega, \tag{6.9a}$$

$$\hat{\tilde{F}}(w;J)^\dagger\Omega\hat{\tilde{A}}(w)\hat{\tilde{F}}(w;J) = \Omega. \tag{6.9b}$$

It should be pointed out that $\hat{M}(J)$ and $M(J)$ give the same pair of real solutions of the SGM-n theory, but $\hat{M}(J) \neq M(J)$ and we have $\hat{H}(J) \neq H(J)$, $\hat{\Gamma}(J) \neq \Gamma(J)$, $\hat{\tilde{\Gamma}}(J) \neq \tilde{\Gamma}(J)$, etc.. Thus, starting from a single solution $M(J)$ of Eqs. (3.3) and (3.1), we obtain two pairs of different ED-complex linear systems (6.5)–(6.9), (3.17)–(3.19), (3.25), and (3.26). Based on solutions $\hat{F}(J)$, $\hat{\tilde{F}}(J)$ of (6.5) and (6.9), we can explicitly construct another infinite set of infinitesimal symmetry transformations of the SGM-n theory as

$$\hat{\delta}\hat{H}(J) = J^2\frac{1}{l}[\hat{F}(l;J)T\hat{F}(l;J)^{-1} - T]\Omega, \tag{6.10a}$$

$$\hat{\tilde{\delta}}\hat{H}(J) = -J^2 s[\hat{\tilde{F}}(s;J)T\hat{\tilde{F}}(s;J)^{-1} - T]\Omega, \tag{6.10b}$$

$$\hat{\Delta}\hat{H}(J) = -J^2\sigma\dot{\hat{F}}(l;J)\hat{F}(l;J)^{-1}\Omega, \tag{6.11a}$$

$$\hat{\tilde{\Delta}}\hat{H}(J) = J^2\epsilon s[s\dot{\hat{\tilde{F}}}(s;J)\hat{\tilde{F}}(s;J)^{-1} + \tfrac{1}{2}]\Omega, \tag{6.11b}$$

which constitute another double representation of the semidirect product of affine $sp(2(\widehat{n+1}),R)$ and Virasoro algebras. Therefore, for each n, the two sets of infinite many double symmetry transformations of the SGM-n theory constructed in this paper constitute a quadruple representation of semidirect product of the Kac-Moody $sp(2(\widehat{n+1}),R)$ and Virasoro algebras. These demonstrate that the SGM-n theories under consideration possess much richer symmetry structures than previously expected.

VII. SUMMARY AND DISCUSSIONS

By using the so-called ED-complex function method,[25] the previously found doubleness symmetry[24] of the SAS SGM-n theories is further exploited and extended in the present paper. A double-complex H-potential $H(J)$ is introduced in (3.8) and the motion equations of the SAS SGM-n theory are written as a double-complex form (3.10). Moreover, we also find that the theories under consideration have more double symmetries which make us be able to introduce an ED-duality mapping (6.1) and establish two pairs of ED-complex HE-type linear systems (3.17)–(3.19), (3.25), (3.26), and (6.5)–(6.9) for each SGM-n theory. We would like to indicate that although Eqs. (3.17), (6.5), (3.25), and (6.8) are, in form, interrelated by $t \leftrightarrow w=1/t$, the analytic properties of $F(t;J)$ $[\hat{F}(t;J)]$ and $\tilde{F}(w;J)$ $[\hat{\tilde{F}}(w;J)]$ as well as the conditions (3.18) and (3.19) [(6.6) and (6.7)] and (3.26) [(6.9)] do not have this interrelation, therefore as whole ED-complex linear systems they are different and give rise to different symmetries of the SGM-n theory. Based on these linear systems, we explicitly construct two sets of double symmetry transformations (4.2), (4.9)–(4.11), (6.10), and (6.11). For any fixed n, these symmetries are verified to constitute quadruple infinite-dim Lie algebras, each of which is a semidirect product of the Kac-Moody $sp(2(\widehat{n+1}),R)$ and Virasoro algebras. These results show that the ED-complex method is necessary and more effective. Some of the results in this paper cannot be obtained by the usual (non-ED-complex) scheme.

Finite symmetry transformations relating to the above infinitesimal ones and soliton solutions of the studied theories need more and further investigations and will be considered in some forthcoming works.

ACKNOWLEDGMENTS

This work was supported by the National Natural Science Foundation of China and the Science Foundation of the Education Department of Liaoning Province, China.

[1] J. Maharana and J. H. Schwarz, Nucl. Phys. B **390**, 3 (1993).
[2] A. Sen, Nucl. Phys. B **447**, 62 (1995); **434**, 179 (1995).
[3] D. V. Gal'tsov, Phys. Rev. Lett. **74**, 2863 (1995).
[4] J. Maharana, Phys. Rev. Lett. **75**, 205 (1995); Phys. Lett. B **348**, 70 (1995); Mod. Phys. Lett. A **11**, 9 (1996).
[5] I. Bakas, Nucl. Phys. B **428**, 374 (1994).
[6] A. K. Biswas, A. Kumar, and K. Ray, Nucl. Phys. B **453**, 181 (1995).
[7] A. Kumar and K. Ray, Phys. Lett. B **358**, 223 (1995).
[8] O. Kechkin and M. Yurova, Phys. Rev. D **54**, 6132 (1996).
[9] D. V. Gal'tsov, *Proceedings of the International Workshop, Leipzig, Germany, 18–22 September 1995*, edited by M. Bordag (Teubner Verlagesellschaft, Stuttgart-Leipzig, 1996), p. 228.
[10] A. Herrera-Aguilar and O. Kechkin, Mod. Phys. Lett. A **12**, 1629 (1997).
[11] A. Herrera-Aguilar and O. Kechkin, Mod. Phys. Lett. A **13**, 1907 (1998).
[12] S. Mizoguchi, Nucl. Phys. B **461**, 155 (1996).
[13] D. V. Gal'tsov, A. A. Garcia, and O. V. Kechkin, Class. Quantum Grav. **12**, 2887 (1995).
[14] D. V. Gal'tsov and O. V. Kechkin, Phys. Lett. B **361**, 52 (1995); Phys. Rev. D **54**, 1656 (1996).
[15] D. V. Gal'tsov and S. A. Sharakin, Phys. Lett. B **399**, 250 (1997).
[16] A. Das, J. Maharana, and A. Melikyan, Phys. Lett. B **518**, 306 (2001); **533**, 146 (2002); Phys. Rev. D **65**, 126001 (2002); Int. J. Mod. Phys. A **19**, 4503 (2004).
[17] N. Barbosa-Cendejas and A. Herrera-Aguilar, Gen. Relativ. Gravit. **35**, 449 (2003).
[18] O. Kechkin and M. Yurova, J. Math. Phys. **39**, 5446 (1998).
[19] I. Hauser and F. J. Ernst, J. Math. Phys. **21**, 1126 (1980).
[20] Y. S. Wu and M. L. Ge, J. Math. Phys. **24**, 1187 (1983).
[21] Y. S. Wu, Phys. Lett. **96A**, 179 (1983).
[22] B. Y. Hou and W. Li, Lett. Math. Phys. **13**, 1 (1987).
[23] W. Li, Phys. Lett. A **129**, 301 (1988).
[24] Y. J. Gao, J. Math. Phys. **44**, 5656 (2003).
[25] Y. J. Gao, Z. Z. Zhong, and Y. X. Gui, J. Math. Phys. **38**, 3155 (1997).
[26] Z. Z. Zhong, J. Math. Phys. **26**, 2589 (1985).
[27] W. Kinnersley, J. Math. Phys. **18**, 1529 (1977).