JOURNAL OF MATHEMATICAL PHYSICS **49**, 083506 (2008)

Inverse scattering method and soliton double solution family for the general symplectic gravity model

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(Received 26 January 2008; accepted 24 June 2008; published online 19 August 2008)

A previously established Hauser–Ernst-type extended double-complex linear system is slightly modified and used to develop an inverse scattering method for the stationary axisymmetric general symplectic gravity model. The reduction procedures in this inverse scattering method are found to be fairly simple, which makes the inverse scattering method applied fine and effective. As an application, a concrete family of soliton double solutions for the considered theory is obtained.
 [DOI: 10.1063/1.2957941]

I. INTRODUCTION

The dimensionally reduced low-energy (super)string effective theories describe various interacting matter fields coupled to gravity and are very significant in theoretical and mathematical physics; the Einstein–Maxwell-dilaton-axion (EMDA) theory (e.g., Refs. 1–5) and its generalized cases containing multiple vector fields[6] are typical and important models of this kind, in which the EMDA theory with six vector fields describes the $N=4$, $D=4$ supergravity.[6]

On the other hand, a series of mathematically possible generalizations of the EMDA theory was suggested by Kechkin and Yurova.[7] These generalized theories, which are different from the EMDA theory with *multiple* vector fields, describe a coupled system of n Abelian vector fields and the symmetric $n\times n$ matrix extensions of the dilaton and Kalb–Ramond fields for $n=1,2,\ldots$ and are called symplectic gravity models (SGMs) by Kechkin and Yurova.[7] We abbreviate these SGMs to SGM-n theories in this paper. Thus the EMDA theory is the case of SGM-1. Some analogies between the SGM-n theories and the reduced vacuum Einstein theory have been noted. However, the mathematical structures of the SGM-n theories are much more complicated and many methods for studying the reduced vacuum gravity (e.g., Refs. 8–12) are no longer applicable. Thus, deeper research and further extended studying methods are needed.

It is undoubted that exact solutions are particularly valuable for understanding the related theories. In the case of general relativity, Belinsky and Zakharov (BZ) (Ref. 8) developed an inverse scattering method (ISM) for solving the two-dimensional vacuum gravity field equations. For the reduced Einstein–Maxwell theory, some ISMs were also proposed.[13] Recently, the BZ ISM has been used to generate (soliton) solutions for some specially restricted systems of the reduced low-energy string effective theories.[14,15] In Ref. 16, by using a modified BZ ISM, from the trivial seed solution and constant "wave function," the author gave some special soliton solutions for the five-dimensional dilaton-axion gravity theory and EMDA theory. In Ref. 17, monodromy transforms and integrability structures for some reduced Einstein field equations and low-energy effective string theories were studied by using some associated linear systems. These results also give some enlightenment for constructing more soliton solutions of the related theories. However, more general application of these schemes to the SGM-n theories still remains a problem. The difficulties have to do with the complicated mathematical structures, such as the fundamental field matrices in the SGM-n theories in general have dimensions greater than 2 and, at the same time,

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must satisfy some nontrivial Riemannian symmetric space (or coset) conditions, etc. [The fundamental field matrix of the SGM-n theory considered in this paper belongs to the symmetric space $\mathrm{Sp}(2(n+1),R)/U(n+1)$.[7]] Therefore, it would be worthwhile to develop some more effective ISM and obtain new exact (soliton) solutions of the SGM-n theories.

In the present paper, we develop an ISM for the stationary axisymmetric (SAS) SGM-n theory to each fixed n. This ISM is different from the above mentioned BZ-like ones, and for arbitrary seed solution and general "wave function" $F(w;J)$ (see below), we can obtain the soliton double solution family of the considered theory. In Ref. 18, two pairs of Hauser–Ernst (HE)–type extended double (ED)–complex linear systems were established for constructing infinitesimal[18] and finite[19] symmetry transformations of the SGM-n theory. Here we show that, with some minor modifications, one of these linear systems is also suitable to the development of ISM for explicitly constructing soliton solutions of the SGM-n theory. These demonstrate that the applicable range of the HE-type linear system is larger than previously expected. As well known, for an ISM the so-called reduction problem is very important and in general is very complicated; so far there is no generally applicable method. However, in this paper we unexpectedly find that, based on the modified HE-type ED-complex linear system, the reduction procedure for the ISM given here is fairly simple.

In Sec. II, some related concepts and notations of the ED-complex function[20] and double motion equations for the SGM-n theories[18] are briefly recalled. In Sec. III, a modified HE-type ED-complex linear system for the SGM-n theory is established. In Sec. IV, an ED dressing transformation is introduced and the associated theorem is proven. In Sec. V, a double ISM for the SGM-n theory is presented. As an application, a family of soliton double solutions for the SGM-n theory is explicitly constructed in Sec. VI. Finally, Sec. VII gives some summary and discussions.

II. ED-COMPLEX FUNCTION AND DOUBLE MOTION EQUATIONS FOR SGM-*n* THEORIES

For later use, here we briefly recall some related concepts and notations of the ED-complex function[20] and double motion equations for the SGM-n theories.[18]

A. ED-complex function

Let i and J denote, respectively, the ordinary and the ED imaginary unit, i.e., $J=j(j^2=-1,j \neq \pm i)$ or $J=\varepsilon(\varepsilon^2=+1,\ \varepsilon \neq \pm 1)$.[20] If a series $\Sigma_{n=0}^{\infty}|a_n|,\ a_n \in \mathbf{C}$ (ordinary complex number) is convergent, then $a(J)=\Sigma_{n=0}^{\infty} a_n\, J^{2n}$ is called an ED ordinary complex number, which corresponds to a pair (a_C,a_H) of ordinary complex number, where $a_C \coloneqq a(J=j)$ and $a_H \coloneqq a(J=\varepsilon)$. When $a(J)$ and $b(J)$ both are ED ordinary complex numbers, $c(J)=a(J)+Jb(J)$ is called an ED-complex number, it corresponds to a pair (c_C,c_H), where $c_C \coloneqq c(J=j)=a_C+jb_C$ and $c_H \coloneqq c(J=\varepsilon)=a_H+\varepsilon b_H$. We denote $a(J) \coloneqq \mathrm{Re}_{\mathrm{ED}}(c(J))$ and $b(J) \coloneqq \mathrm{Im}_{\mathrm{ED}}(c(J))$. If $a(J)$ and $b(J)$ are both real, we call them double real and call the corresponding $c(J)$ simply a double-complex number.[21]

All ED-complex numbers with usual addition and multiplication constitute a commutative ring. Corresponding to the two imaginary units J and i in this ring, we have two complex conjugations: ED-complex conjugation "$\star$" and ordinary complex conjugation "–,"

$$c(J)^{\star} \coloneqq a(J) - Jb(J), \quad \overline{c(J)} \coloneqq \overline{a(J)} + J\overline{b(J)}. \tag{2.1}$$

These imply that $J^{\star}=-J$, $\bar{J}=J$, $i^{\star}=i$, and $\bar{i}=-i$. If $a(J)$ and $b(J)$ are ED ordinary complex functions of some ordinary complex variables $z_1,\ldots,z_n$, then $c(z_1,\ldots,z_n;J)=a(z_1,\ldots,z_n;J)+Jb(z_1,\ldots,z_n;J)$ is called an ED-complex function. We say $c(z_1,\ldots,z_n;J)$ to be continuous, analytical, etc., if $a(z_1,\ldots,z_n;J)$ and $b(z_1,\ldots,z_n;J)$ both, as ordinary complex functions, have the same properties. For an ED-complex matrix $W(J)$, we define

$$W(J)^{+} \coloneqq [W(J)^{\star}]^{\top}, \tag{2.2}$$

where "$\top$" denotes the transposition.

B. Double motion equations of the SGM-n theories

The actions of SGM-n theories in four dimensions are[7]

$$S=\int\left\{-R+\mathrm{Tr}\left[\frac{1}{2}(\partial pp^{-1})^2-pFF^\top+\frac{1}{3}(pH)^2\right]\right\}\sqrt{-g}d^4x, \tag{2.3}$$

where $g_{\mu\nu}$ is the metric (signature $+---$, $\mu,\nu=0,1,2,3$), R is the Ricci scalar, $g=\det(g_{\mu\nu})$, p is a symmetric $n\times n$ matrix with scalar field components (for the EMDA case, $p=e^{-2\phi}$, where ϕ is the dilaton field), and

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu,\quad H_{\mu\nu\lambda}=\partial_\mu B_{\nu\lambda}-\tfrac{1}{2}(A_\mu F_{\nu\lambda}^\top+F_{\nu\lambda}A_\mu^\top)+\text{cyclic}. \tag{2.4}$$

A_μ is an $n\times 1$ column of Abelian vector fields and $B_{\mu\nu}$ is an extension of the usual Kalb–Ramond tensor field; here each Lorentz component of $B_{\mu\nu}$ is a symmetric $n\times n$ matrix and among these matrices there are relations $B_{\mu\nu}=-B_{\nu\mu}$. The SGM-n action (2.3) gives the pure Einstein and the EMDA theory, respectively, when $n=0$ and $n=1$, and provides their mathematical generalization for an arbitrary non-negative integer $n>1$.

In the SAS case, the four-dimensional space-time line element can be written as[22]

$$ds^2=f_{AB}dx^Adx^B-e^\gamma\delta_{LN}dx^Ldx^N\quad(A,B=0,1,L,N=2,3), \tag{2.5}$$

and f_{AB} can be parametrized as

$$f_{AB}=\begin{pmatrix} f & -f\omega\\ -f\omega & f\omega^2-\rho^2f^{-1}\end{pmatrix}. \tag{2.6}$$

The set of nontrivial SGM-n dynamical quantities also contains two Lorentzian components A_0, A_1 of the $n\times 1$ column four-potential A_μ, one nontrivial Lorentzian component B_{01} of the matrix extended field $B_{\mu\nu}$ and the $n\times n$ matrix field p; all of these fields are assumed to depend only on x^2, x^3. Introducing $(n+1)\times(n+1)$ symmetric real matrices P and Q by

$$P=\begin{pmatrix} f-2A_0^\top pA_0 & -\sqrt{2}A_0^\top p\\ -\sqrt{2}pA_0 & -p\end{pmatrix},$$

$$Q=\begin{pmatrix} \omega & -\sqrt{2}(A_1+\omega A_0)^\top\\ -\sqrt{2}(A_1+\omega A_0) & (A_1+\omega A_0)A_0^\top+A_0(A_1+\omega A_0)^\top-B_{01}\end{pmatrix}. \tag{2.7}$$

then the essential dynamical equations of the SAS SGM-n theory can be written as[7]

$$d(\rho^{-1}P^*dQP)=0,\quad d(\rho^*dPP^{-1}+\rho^{-1}P^*dQPQ)=0, \tag{2.8}$$

and $\rho=\rho(x^2,x^3)>0$ is a harmonic function in two dimensions $\{x^2,x^3\}$, where the notations of differential form are adopted, and "$*$" is the dual operation of two-dimensional Euclidian space.

As pointed out in Refs. 18 and 23, the SAS SGM-n theories possess the so-called doubleness symmetry such that for any non-negative n, we can introduce $(n+1)\times(n+1)$ double-real symmetric matrices $P(J)$ and $Q(J)$ and define a double-real $2(n+1)\times 2(n+1)$ matrix function $M(J)=M(x^2,x^3;J)$ as

$$M(J)=\begin{pmatrix} P(J) & -P(J)Q(J)\\ -Q(J)P(J) & Q(J)P(J)Q(J)+J^2\rho^2P(J)^{-1}\end{pmatrix}, \tag{2.9}$$

and the motion equations [Eq. (2.8)] can be extended to a double formulation

$$d(\rho^{-1}M(J)\eta^*dM(J))=0, \tag{2.10}$$

with conditions

$$M(J)^{\top} = M(J), \tag{2.11a}$$

$$M(J)\eta M(J) = J^2\rho^2\eta, \tag{2.11b}$$

$$\eta := \begin{pmatrix} 0 & I_{n+1} \\ -I_{n+1} & 0 \end{pmatrix}, \tag{2.11c}$$

where I_{n+1} is the $n+1$-dimensional unit matrix. If a solution of Eqs. (2.10), (2.11a), (2.11b), and (2.11c) is known, then by the decomposition (2.9), we can obtain real solutions of the SGM-n theory *in pairs* as follows:

$$(P,Q) = (P_C,Q_C), \tag{2.12a}$$

$$(\hat{P},\hat{Q}) = (T(P_H),V_{P_H}(Q_H)), \tag{2.12b}$$

where the transformations T, V are defined by

$$T:\ \ P \to T(P) = \rho P^{-1},$$

$$V:\ \ P,Q \to V_P(Q) = \int \rho^{-1}P(\partial_{x^3}Q)Pdx^2 - \rho^{-1}P(\partial_{x^2}Q)Pdx^3, \tag{2.13}$$

and the existence of $V_{P_H}(Q_H)$ is ensured by the $J=\epsilon$ case of Eq. (2.10).

III. MODIFIED HE-TYPE ED-COMPLEX LINEAR SYSTEM

Equation (2.10) implies that we can introduce a double-real $2(n+1)\times 2(n+1)$ matrix twist potential $K(x^2,x^3;J)$ by $dK(J)=-\rho^{-1}M(J)\eta{}^*dM(J)$, then from (2.11a), (2.11b), and (2.11c) and harmony of $\rho(x^2,x^3)$ we can obtain $K(J)-K(J)^{\top}=-2J^2z\eta$ with the real field $z=z(x^2,x^3)$ introduced by ${}^*d\rho=dz$. Thus, if we define a double-complex H-potential

$$H(J) := M(J) + JK(J) \tag{3.1}$$

and denote $\Omega := J\eta$, then the equations about $K(J)$ and $M(J)$ can be written together as

$$2(z+\rho^*)dH(J) = (H(J)+H(J)^+)\Omega dH(J). \tag{3.2}$$

Now we introduce an ordinary complex parameter w and define

$$A(w;J) := w - (H(J)+H(J)^+)\Omega, \tag{3.3}$$

$$\Gamma(w;J) := \Lambda^{-1}(w)dH(J), \tag{3.4}$$

$$\Lambda(w) := w - 2(z+\rho^*), \quad \Lambda^{-1}(w) = \lambda(w)^{-2}[w-2(z-\rho^*)], \tag{3.5}$$

$$\lambda(w) := [(w-2z)^2 + (2\rho)^2]^{1/2}, \quad \lambda(w)|_{w=\infty} = w. \tag{3.6}$$

Then from Eqs. (3.2)–(3.4), we can obtain

$$d\Gamma(w;J) = \Gamma(w;J)\Omega \wedge \Gamma(w;J), \tag{3.7}$$

where "$\wedge$" denotes the exterior product of differential forms. Equation (3.7) is just the complete integrability condition of the following ED-complex linear differential equation

$$dF(w;J) = \Gamma(w;J)\Omega F(w;J), \tag{3.8}$$

where $F(w;J)=F(x^2,x^3,w;J)$ is a $2(n+1)\times 2(n+1)$ matrix ED-complex function of x^2, x^3, and w. Moreover, from the definitions and properties of ρ and z, without loss of generality, we shall redefine coordinates by setting $x^2=\rho$, $x^3=z$ for simplicity in the following.

Equation (3.8) does not define $F(w;J)$ uniquely; we shall impose some subsidiary conditions being consistent with the above equations and the requirement that $F(w;J)$ be holomorphic in a neighborhood of $w=\infty$. To this end, we note that if $F(w;J)$ is expanded in power of w^{-1} around $w=\infty$ as

$$F(w;J) = F^{(0)}(\rho,z;J) + w^{-1}F^{(1)}(\rho,z;J) + \cdots, \tag{3.9}$$

then from (3.2), (3.8), and the relation $2\Lambda^{-1}(w)dz=-\lambda(w)^{-1}d\lambda(w)$, we have

$$dF^{(0)}(J) = 0, \quad d[F^{(1)}(J) - H(J)\Omega F^{(0)}(J)] = 0,$$

$$d[\lambda(w)F(w;J)^{\top}\Omega F(w;J)] = 0, \quad d[F(w;J)^{+}\Omega A(w;J)F(w;J)] = 0,$$

where the ED-Hermitian conjugation "+" is defined by (2.2). These equations and (3.8) determine $F(w;J)$ up to right multiplication by an arbitrary nondegenerate $2(n+1)\times 2(n+1)$ matrix function of w, so we can use this freedom and choose the integral constants consistently such that

$$F^{(0)}(J) = I, \tag{3.10a}$$

$$F^{(1)}(J) = H(J)\Omega, \tag{3.10b}$$

$$\lambda(w)F(w;J)^{\top}\Omega F(w;J) = w\Omega, \tag{3.11a}$$

$$F(w;J)^{+}\Omega A(w;J)F(w;J) = w\Omega, \tag{3.11b}$$

where I is the $2(n+1)$-dimensional unit matrix. Equations (3.8), (3.9), (3.10a), (3.10b), (3.11a), and (3.11b) are essentially the HE-type ED-complex linear system for $F(t;J)$ in Ref. 18 with t being replaced by w^{-1}. Besides, from definition (3.3) we see that

$$A(w;J) \text{ is a linear function of } w. \tag{3.12}$$

We call (3.8), (3.9), (3.10a), (3.10b), (3.11a), (3.11b), and (3.12) a modified HE-type ED-complex linear system for the SGM-n theory.

Now we give some other useful relations. Introducing ordinary complex coordinates $\zeta := z + i\rho$, $\xi := z - i\rho$, Eq. (3.2) can be rewritten as

$$[2\zeta - (H(J) + H(J)^{+})\Omega]\partial_{\zeta}H(J) = 0, \tag{3.13a}$$

$$[2\xi - (H(J) + H(J)^{+})\Omega]\partial_{\xi}H(J) = 0. \tag{3.13b}$$

From the definition of $A(w;J)$ in (3.3), Eqs. (3.13a) and (3.13b) may be written as

$$A(w = 2\zeta;J)\partial_{\zeta}H(J) = 0, \tag{3.14a}$$

$$A(w = 2\xi;J)\partial_{\xi}H(J) = 0. \tag{3.14b}$$

Moreover, for any $A(w;J)$ satisfying Eqs. (3.11a) and (3.11b), we have

$$A(w;J)^{\top}\Omega A(w;J) = \lambda^2(w)\Omega. \tag{3.15}$$

IV. ED DRESSING TRANSFORMATION

As can be seen in the following, by virtue of the solution $F(w;J)$ of linear systems (3.8), (3.9), (3.10a), (3.10b), (3.11a), (3.11b), and (3.12), we can explicitly construct new double solutions of the SGM-n theory. At first, from definitions (3.1) and (3.3)–(3.6), we may consistently choose the ED-complex matrix functions $F(w;J)$ as

$$\overline{F(w;J)} = F(\bar{w};J) \tag{4.1}$$

in order to ensure the reality of the new solutions. We shall take this choice in the following.

Assuming that we have a solution $M_0(J)$ (seed solution) of Eqs. (2.10), (2.11a), (2.11b), and (2.11c), then we can obtain the corresponding $H_0(J)$, $A_0(w;J)$, and solution $F_0(w;J)$ of (3.8), (3.9), (3.10a), (3.10b), (3.11a), and (3.11b). Now we take a dressing transformation

$$F(w;J) = \chi(w;J)F_0(w;J), \tag{4.2}$$

where $\chi(w;J)=\chi(\rho,z,w;J)$ is a $2(n+1)\times 2(n+1)$ matrix ED-complex function of ρ, z, and w. Condition (4.1) and the requirement that $F(w;J)$ be also a solution of (3.8), (3.9), (3.10a), (3.10b), (3.11a), and (3.11b) [for some $H(J)$ and the associated $A(w;J)$] imply

$$\overline{\chi(w;J)} = \chi(\bar{w};J), \tag{4.3a}$$

$$\chi(\infty;J) = I, \tag{4.3b}$$

$$d\chi(w;J) = \Lambda^{-1}(w)[dH(J)\Omega\chi(w;J) - \chi(w;J)dH_0(J)\Omega], \tag{4.4}$$

$$\chi^{\top}(w;J)\Omega\chi(w;J) = \Omega, \tag{4.5a}$$

$$\chi^{+}(w;J)\Omega A(w;J)\chi(w;J) = \Omega A_0(w;J). \tag{4.5b}$$

Noticing (4.3a) and (4.3b) and expressing the expansion of $\chi(w;J)$ in the neighborhood of $w=\infty$ as

$$\chi(w;J) = I + w^{-1}\chi^{(1)}(\rho,z;J) + \cdots, \tag{4.6}$$

then we have the following.

Theorem 1: If $H_0(J)$ is an H-potential for some known $M_0(J)$ and $\chi(w;J)$ satisfies Eqs. (4.3a), (4.3b), (4.4), (4.5a), and (4.5b) with $A(w;J)$ fulfilling (3.12), then

$$H(J) := H_0(J) - J^2\chi^{(1)}(J)\Omega \tag{4.7}$$

is a double-complex H-potential of the SAS SGM-n theory, i.e., $M(J)=\mathrm{Re}_{\mathrm{ED}}(H(J))$ is a solution of Eq. (2.10) with conditions (2.11a) and (2.11b).

Proof: First, from (4.3a) and (4.6), $H(J)$ in (4.7) is double complex (cf. Sec. II A for the meaning of "double complex"). Moreover, Eqs. (4.5b), (4.6), and (3.12) give

$$A(w;J) = w - (H(J) + H(J)^{+})\Omega, \tag{4.8}$$

with $H(J)$ defined by (4.7). In terms of coordinates ζ and ξ, Eq. (4.4) can be written as

$$\partial_\zeta H(J)\Omega = \chi(w;J)\partial_\zeta H_0(J)\Omega\chi^{-1}(w;J) - (w-2\zeta)\partial_\zeta\chi(w;J)\chi^{-1}(w;J),$$

$$\partial_\xi H(J)\Omega = \chi(w;J)\partial_\xi H_0(J)\Omega\chi^{-1}(w;J) - (w-2\xi)\partial_\xi\chi(w;J)\chi^{-1}(w;J).$$

Thus, if $\chi(w;J)$ and $\chi^{-1}(w;J)$ are not singular at $w=2\zeta$ and $w=2\xi$ (for the soliton transformation in the following, these conditions are automatically fulfilled), the above two equations give

$$\partial_\zeta H(J)\Omega = \chi(w=2\zeta;J)\partial_\zeta H_0(J)\Omega\chi^{-1}(w=2\zeta;J), \tag{4.9a}$$

$$\partial_\xi H(J)\Omega = \chi(w=2\xi;J)\partial_\xi H_0(J)\Omega\chi^{-1}(w=2\xi;J). \tag{4.9b}$$

Since $H_0(J)$ and $A_0(w;J)$ satisfy Eqs. (3.14a) and (3.14b), from Eqs. (4.5b), (3.14a), and (3.14b) [for $H_0(J)$, $A_0(w;J)$] we have

$$A(w=2\zeta;J)\partial_\zeta H(J) = 0, \tag{4.10a}$$

$$A(w=2\xi;J)\partial_\xi H(J) = 0. \tag{4.10b}$$

These say that $H(J)$ given by (4.7) satisfies Eq. (3.2).

Now, Eqs. (4.4) and (4.5a) give out

$$dH(J) = \chi(w;J)dH_0(J)\chi^\top(w;J) - J^2\Lambda(w)d\chi(w;J)\Omega\chi^\top(w;J). \tag{4.11}$$

Since for a known $M_0(J)$ we have $H_0(J)-H_0(J)^\top=-2J^2z\Omega$ by definitions of $H_0(J)$ and z, Eq. (4.11) gives

$$dH(J) - dH(J)^\top = -2J^2dz\chi(w;J)\Omega\chi^\top(w;J) - J^2\Lambda(w)d[\chi(w;J)\Omega\chi^\top(w;J)].$$

Thus using Eq. (4.5a) again we obtain

$$H(J) - H(J)^\top = -2J^2z\Omega \tag{4.12}$$

by choosing some suitable integral constant. Equation (4.12) immediately implies that $M(J) := \mathrm{Re}_{\mathrm{ED}}(H(J))$ satisfies (2.11a). Moreover, by (3.11a), (3.11b), (4.2), (4.5a), and (4.5b) [for $F_0(w;J)$ and $A_0(w;J)$], it follows that $A(w;J)$ satisfies Eq. (3.15), then from Eqs. (4.8) and (4.12) we can see that $M(J) := \mathrm{Re}_{\mathrm{ED}}(H(J))$ satisfies condition (2.11b). Finally, with conditions (2.11a) and (2.11b), Eq. (3.2) implies Eq. (2.10).

V. DOUBLE ISM

For an ISM of solving some nonlinear equations, the so-called reduction problem is very important. However, the reduction procedures vary with the equations and the associated linear systems and, in general, are very difficult; so far there is no generally applicable method. In this section, based on the modified HE-type ED-complex linear system and ED dressing transformation given above, we develop a double ISM for the SGM-n theory and unexpectedly find that the reduction procedures for this ISM are fairly simple.

To construct pure N-soliton solutions of the SAS SGM-n theory, we take the following ansatz for $\chi(w;J)$ and $\chi^{-1}(w;J)$:

$$\chi(w;J) = I + \sum_{k=1}^{N}\frac{R_k(J)}{w-\mu_k}, \tag{5.1a}$$

$$\chi^{-1}(w;J) = I + \sum_{k=1}^{N}\frac{S_k(J)}{w-\nu_k}, \tag{5.1b}$$

where [owing to (4.3a)] the poles μ_k, ν_k are real, $R_k(J)=R_k(\rho,z;J)$, $S_k(J)=S_k(\rho,z;J)$ are double-complex $2(n+1)\times 2(n+1)$ matrix functions, and they are all independent of w. The relations $\chi(w;J)\chi^{-1}(w;J)=\chi^{-1}(w;J)\chi(w;J)=I$ imply

$$R_k(J)\chi^{-1}(\mu_k;J) = \chi^{-1}(\mu_k;J)R_k(J) = 0, \quad S_k(J)\chi(\nu_k;J) = \chi(\nu_k;J)S_k(J) = 0. \tag{5.2}$$

Thus $R_k(J)$ and $S_k(J)$ are singular matrices. In the following, we take $R_k(J)$ and $S_k(J)$ to be of rank 1[1] and write them as

$$R_k(J) = m_k(J)n_k(J), \quad S_k(J) = p_k(J)q_k(J), \quad k = 1, \ldots, N \quad (\text{no sum in } k), \tag{5.3}$$

where $m_k(J)=m_k(\rho,z;J)$ and $p_k(J)=p_k(\rho,z;J)$ are $2(n+1)$-dimensional column vectors, and $n_k(J)=n_k(\rho,z;J)$ and $q_k(J)=q_k(\rho,z;J)$ are $2(n+1)$-dimensional row vectors. By (5.1a), (5.1b), and (5.3), Eq. (5.2) gives

$$n_k(J) + \sum_{l=1}^{N} \frac{n_k(J)p_l(J)}{\mu_k - \nu_l} q_l(J) = 0, \quad p_k(J) - \sum_{l=1}^{N} \frac{n_l(J)p_k(J)}{\mu_l - \nu_k} m_l(J) = 0, \tag{5.4}$$

where $n_k(J)p_l(J)$ is a scalar product of row vector $n_k(J)$ and column vector $p_l(J)$. Equation (5.4) enables us to express all of the vector functions $m_k(J)$'s and $q_k(J)$'s in terms of the set of $2N$ vector functions $n_k(J)$'s and $p_k(J)$'s,

$$m_k(J) = \sum_{l=1}^{N} G_{lk}^{-1}(J)p_l(J), \quad q_k(J) = -\sum_{l=1}^{N} G_{kl}^{-1}(J)n_l(J), \tag{5.5}$$

where $G_{kl}^{-1}(J)$'s are elements of the inverse of matrix $G(J)=\{G_{lk}(J)\}_{l,k=1}^{N}$ with

$$G_{lk}(J) \coloneqq \frac{n_l(J)p_k(J)}{\mu_l - \nu_k}. \tag{5.6}$$

Therefore, we only need to determine $n_k(J),p_k(J)$ and μ_k,ν_k in the following.

The functions $R_k(J)$, $S_k(J)$, μ_k, and ν_k must also obey Eqs. (4.4), (4.5a), and (4.5b). Writing Eq. (4.4) as

$$\begin{aligned}\Lambda^{-1}(w)dH(J)\Omega &= [\chi(w;J)\Gamma_0(w;J)\Omega + d\chi(w;J)]\chi^{-1}(w;J) = \chi(w;J)[\Gamma_0(w;J)\Omega\chi^{-1}(w;J) \\ &\quad - d\chi^{-1}(w;J)],\end{aligned} \tag{5.7}$$

with $\Gamma_0(w;J)=\Lambda^{-1}(w)dH_0(J)$, then the substitution of Eqs. (5.1a), (5.1b), and (5.3) into (5.7) shows that μ_k,ν_k should all be real constants and $n_k(J),p_k(J)$ satisfy the equations

$$dn_k(J) + n_k(J)\Gamma_0(\mu_k;J)\Omega = 0, \quad dp_k(J) - \Gamma_0(\nu_k;J)\Omega p_k(J) = 0. \tag{5.8}$$

Comparing Eq. (5.8) with (3.8) [for seed solution $H_0(J)$], we see that the solutions of Eq. (5.8) can be written as

$$n_k(J) = n_k^{(0)}(J)F_0^{-1}(\mu_k;J), \quad p_k(J) = F_0(\nu_k;J)p_k^{(0)}(J), \tag{5.9}$$

where $n_k^{(0)}(J)$ and $p_k^{(0)}(J)$ are $2(n+1)$-dimensional double-complex constant row and column vectors, respectively. Further, when substituting Eqs. (5.1a), (5.1b), (5.3), (5.5), and (5.9) into (4.5a) and (4.5b), noticing (3.12) and by some algebraic calculations, we surprisingly find that conditions (4.5a) and (4.5b) can be simultaneously fulfilled provided that we take

$$\nu_k = \mu_k, \tag{5.10a}$$

$$n_k^{(0)}(J) = p_k^{(0)}(J)^{\top}\eta, \tag{5.10b}$$

with $p_k^{(0)}(J)$'s being double real and η being defined by Eq. (2.11c).

[1] This is not the only possible choice. In general, the rank r of $R_k(J)$ can be taken as $1 \le r \le 2n+1$, and similarly for S_k.

It must be also noted that from (5.9), (5.10a), and (5.10b) and the relation $p_k^{(0)}(J)^\top \eta p_k^{(0)}(J) = 0$ for any $2(n+1)$-dimensional column vector $p_k^{(0)}(J)$, the expressions of $G_{kk}(J)$ ($k=1,2,\ldots,N$) in (5.6) are indefinite and need to be further determined. From the function theory of complex variable, we can have

$$G_{kk}(J) = -p_k^{(0)}(J)^\top \eta F_0^{-1}(\mu_k;J)[\partial_w F_0(w;J)]_{w=\mu_k} p_k^{(0)}(J), \quad k=1,2,\ldots,N. \tag{5.11}$$

So far, we have obtained $\chi(w;J)$ in terms of $F_0(w;J)$; from Theorem 1 and noticing relation (3.11a) for $F_0(w;J)$, we can get the new N-soliton double-complex H-potential $H_N(J)$ by formula (4.7) with

$$\chi_N^{(1)}(J) = \sum_{l,k=1}^{N} \frac{\lambda(\mu_k)}{\mu_k} G_{lk}^{-1}(J) F_0(\mu_l;J) p_l^{(0)}(J) p_k^{(0)}(J)^\top F_0^\top(\mu_k;J)\eta, \tag{5.12}$$

where

$$G_{lk}(J) = \frac{\lambda(\mu_l)}{\mu_l(\mu_l-\mu_k)} p_l^{(0)}(J)^\top F_0^\top(\mu_l;J)\eta F_0(\mu_k;J) p_k^{(0)}(J), \quad k \neq l, \ k, \ l = 1,\ldots,N,$$

$$G_{kk}(J) = -\frac{\lambda(\mu_k)}{\mu_k} p_k^{(0)}(J)^\top F_0^\top(\mu_k;J)\eta[\partial_w F_0(w;J)]_{w=\mu_k} p_k^{(0)}(J), \quad k=1,\ldots,N, \tag{5.13}$$

and $p_k^{(0)}(J)$'s are arbitrary $2(n+1)$-dimensional double-real constant column vectors.

VI. APPLICATION: SOLITON DOUBLE SOLUTION FAMILY OF SGM-*n* THEORY

To illustrate the ISM given above, now we concretely construct a family of soliton solutions for the SGM-n theory, to which we choose double Minkowsky space-time as a seed solution. The corresponding $M_0(\rho,z;J)$, $H_0(\rho,z;J)$, and $F_0(\rho,z,w;J)$ are

$$M_0(J) = \begin{pmatrix} \widetilde{I}_{n+1} & 0 \\ 0 & J^2\rho^2\widetilde{I}_{n+1} \end{pmatrix}, \quad H_0(J) = \begin{pmatrix} \widetilde{I}_{n+1} & -2J^3 z I_{n+1} \\ 0 & J^2\rho^2\widetilde{I}_{n+1} \end{pmatrix}; \quad \widetilde{I}_{n+1} := \begin{pmatrix} 1 & 0 \\ 0 & -I_n \end{pmatrix}, \tag{6.1}$$

$$F_0(\rho,z,w;J) = \frac{1}{\lambda(w)} \begin{pmatrix} wI_{n+1} & J\widetilde{I}_{n+1} \\ \dfrac{J^3 w}{2}[w-2z-\lambda(w)]\widetilde{I}_{n+1} & \dfrac{1}{2}([w-2z+\lambda(w)]I_{n+1}) \end{pmatrix}. \tag{6.2}$$

Now for simplifying the notations, we write

$$p_k^{(0)}(J) = \begin{pmatrix} a_k(J) \\ b_k(J) \end{pmatrix} \ [a_k(J), b_k(J)]$$

are both arbitrary double-real constant $(n+1)$-dimensional column vectors and introduce

$$\lambda_k = \lambda_k(\rho,z) := \lambda(\mu_k), \quad T_{(\pm)k} := \mu_k - 2z \pm \lambda_k, \quad \widetilde{a}_k := \widetilde{I}_{n+1} a_k, \quad \widetilde{b}_k := \widetilde{I}_{n+1} b_k.$$

Then from Eqs. (4.7), (5.12), (5.13), and (6.2), we can obtain the corresponding soliton double-complex H-potential family as follows:

$$H_N(\rho,z;J) = H_0(J) + J^3 \sum_{l,k=1}^{N} \widetilde{G}_{lk}^{-1}(J) u_l(J) u_k^\top(J), \ \ N=1,2,\ldots, \tag{6.3}$$

where

$$\widetilde{G}_{lk}(J) = \frac{1}{\mu_l - \mu_k} u_l^{\top}(J)\eta u_k(J), \quad k \neq l, \quad k,l = 1, \ldots, N,$$

$$\widetilde{G}_{kk}(J) = -\frac{1}{\lambda_k^2} u_k^{\top}(J)\eta v_k(J), \quad k = 1, \ldots, N, \tag{6.4}$$

and

$$u_k(J) = \begin{pmatrix} \mu_k a_k(J) + J\widetilde{b}_k(J) \\ \frac{\mu_k}{2} J^3 T_{(-)k}\widetilde{a}_k(J) + \frac{T_{(+)k}}{2} b_k(J) \end{pmatrix},$$

$$v_k(J) = \begin{pmatrix} [(2\rho)^2 - 2z(\mu_k - 2z)]a_k(J) + (2z - \mu_k)J\widetilde{b}_k(J) \\ \frac{J^3}{2}(T_{(-)k}\lambda_k^2 + 4\mu_k\rho^2)\widetilde{a}_k(J) + 2\rho^2 b_k(J) \end{pmatrix}. \tag{6.5}$$

Thus, we obtain a concrete soliton double solution family $\{M_N(J) = \mathrm{Re}_{\mathrm{ED}}(H(J)), \quad N = 1, 2, \ldots\}$. By using formulas (2.12a) and (2.12b), for each $M_N(J)$ we can obtain a pair of real solutions of the SGM-n theory.

VII. SUMMARY AND DISCUSSIONS

A previously established HE-type ED-complex linear system[18] is slightly modified and used to develop a double ISM for the SGM-n theory. We show that the reduction procedures for this ISM are fairly simple, which makes the ISM applied fine and effective. As an application, we obtain a concrete family of soliton double solutions for the SAS SGM-n theory. The discussions in this paper are applicable to some nonlinear σ-models on other symmetric spaces.

ACKNOWLEDGMENTS

This work was supported by the National Natural Science Foundation of China and the Science Foundation of the Education Department of Liaoning Province, China.

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